EXERCISE – III HINTS & SOLUTIONS

Sol.1 $A_{3\times 2} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$

Sol.2 Given $\begin{bmatrix} x-y & 1 & z \\ 2x-y & 0 & w \end{bmatrix} = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

If two matrices are equal then their corresponding elements are equal

$\therefore$ $x - y = -1$, $z = 4$, $2x - y = 0$, $w = 5$

Hence $x = 1$, $y = 2$, $z = 4$, $w = 5$

Sol.3 Given $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix}$

order of A = 3 × 2; order of B = 2 × 3

so AB and BA exist but AB ≠ BA

because order of AB ≠ order of BA

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix} = \begin{bmatrix} 18 & -11 & 10 \\ -16 & 47 & 10 \\ 62 & -23 & 42 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 49 & 24 \\ -7 & 58 \end{bmatrix}$$

Sol.4 If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$

where k is any +ve integer

$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then $A^1 = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix}_{k=1}$

Also $A^2 = AA = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 9-4 & -12+4 \\ 3-1 & -4+1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1+2.2 & -4.2 \\ 2 & 1-2.2 \end{bmatrix}_{k=2}$$

Now assume that $A^k = \begin{bmatrix} 1+2k & -12+4 \\ k & 1-2k \end{bmatrix}$

$\therefore$ $A^{k+1} = AA^k = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$

$$= \begin{bmatrix} 3+2k & -4k-4 \\ 1+k & -2k-1 \end{bmatrix} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

we observe that our assumption is true for k = k + 1 and it was true when k = 1 or 2. Hence it is true for all +ve integral values of k.

Sol.5 **(i)** (adj adj A) = $|A|^{n-2}$ A

As we know that

A(adj A) = |A| I

Therefore, (adj A) (adj adj A) = |adj A| I

$\Rightarrow$ A(adj A) (adj adj A) = $|A|^{n-1}$ (AI)

[$\because$ $|A|^{n-1}$ is a scalar]

$\Rightarrow$ |A| I (adj adj A) = $|A|^{n-1}$ A

$\Rightarrow$ adj adj A = $|A|^{n-2}$ A [$\because$ |A| is a scalar]

(ii) As we know that, $|\text{adj adj A}| = |A|^{(n-1)^2}$

Therefore,

$\Rightarrow$ $|\text{adj adj adj A}| = |\text{adj A}|^{(n-1)^2}$

$\Rightarrow$ $|\text{adj adj adj A}| = \left(|A|^{(n-1)}\right)^{(n-1)^2}$

$\Rightarrow$ $|\text{adj adj adj A}| = |A|^{(n-1)^3}$

Sol.6 Given $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and $A^2 + aA + bI = 0$ ....(1)

Since A is a square matrix

We have $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 1$

$\therefore$ the characteristic equation of A is $\lambda^2 - 4\lambda + 1 = 0$

By the cayley-hamilton theorem $A^2 - 4A + I = 0$ ....(2)

$\Rightarrow$ a = (–4); b = 1

Verification of (2) we have

$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 4 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 12 & 8 \\ 4 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence cayley – Hamiltion theorem is verified

Now we shall compute A^{-1}

Multiplying (2) by A^{-1} we get $A - 4I + A^{-1} = 0$

$\therefore$ $A^{-1} = -(A - 4I) = \left(\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}$$

Sol.7 Given, $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Now adj $B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ and $|B| = 1$

then $B^{-1} = \dfrac{\text{adj}\, B}{|B|} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

$\therefore \quad (AB)^{-1} = B^{-1} A^{-1}$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Sol.8 Given, $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

and $AB - CD = 0$

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$AB - CD = 0$

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} - \begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3-2a-5c & -2b-5d \\ 43-3a-8c & 22-3b-8d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \quad 3 - 2a - 5c = 0 \Rightarrow 2a + 5c = 3$

$43 - 3a - 8c = 0 \Rightarrow 3a + 8c = 43$

$-2b - 5d = 0 \Rightarrow 2b + 5d = 0$

$22 - 3b - 8d = 0 \Rightarrow 3b + 8d = 22$

Hence $a = -191$, $b = -110$, $c = 77$, $d = 44$

so $\quad D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$

Sol.9 Given A and B are two square matrices such that

AB = A & BA = B

To prove : A & B are idempotent.

Now, AB = A & BA = B

$\Rightarrow A^{-1} A B = A^{-1} A$ & $B^{-1} B = B^{-1} B$

$\Rightarrow IB = I$ & $IA = I$

$\Rightarrow B = I$ & $A = I$

we know that I is always idempotent matrix so A and B are also idempotent

Sol.10 Given $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

since $A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

$$= \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

therefore A is a nilpotent matrix

Sol.11 Given $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

Now, $\dfrac{1}{2}(A - A' + I) = B$(say)

$$= \frac{1}{2}\left\{\begin{bmatrix} -2 & 3 & 4 \\ 5 & -4 & -3 \\ 7 & 2 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 5 & 7 \\ 3 & -4 & 2 \\ 4 & -3 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & -5 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 & -3/2 \\ 1 & 1/2 & -5/2 \\ 3/2 & 5/2 & 1/2 \end{bmatrix} = B$$

Now find B^{-1} by using elementary transformation we have, $BB^{-1} = I$

$$\Rightarrow \begin{bmatrix} 1/2 & -1 & -3/2 \\ 1 & 1/2 & -5/2 \\ 3/2 & 5/2 & 1/2 \end{bmatrix} B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By applying $(R_1 \leftrightarrow R_2)$, we get

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & -5/2 \\ 1/2 & -1 & -3/2 \\ 3/2 & 5/2 & 1/2 \end{bmatrix} B^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By applying $\begin{pmatrix} R_2 \to R_2 - \frac{1}{2}R_1 \\ R_3 \to R_3 - \frac{3}{2}R_1 \end{pmatrix}$, we get

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & -5/2 \\ 0 & -5/4 & -1/4 \\ 0 & 7/4 & 17/4 \end{bmatrix} B^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1/2 & 0 \\ 0 & -3/2 & 1 \end{bmatrix}$$

By applying $R_2 \to -\frac{4}{5} R_2$, we get

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & -5/2 \\ 0 & 1 & 1/5 \\ 0 & 7/4 & 17/4 \end{bmatrix} B^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 2/5 & 0 \\ 0 & -3/2 & 1 \end{bmatrix}$$

By applying $\begin{pmatrix} R_1 \to R_1 - \frac{1}{2} R_2 \\ R_3 \to R_3 - \frac{7}{4} R_2 \end{pmatrix}$, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & -13/5 \\ 0 & 1 & 1/5 \\ 0 & 0 & 39/10 \end{bmatrix} B^{-1} = \begin{bmatrix} 2/5 & 4/5 & 0 \\ -4/5 & 2/5 & 0 \\ 7/5 & -11/5 & 1 \end{bmatrix}$$

By applying $R_3 \to \frac{10}{39} R_3$, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & -13/5 \\ 0 & 1 & 1/5 \\ 0 & 0 & 1 \end{bmatrix} B^{-1} = \begin{bmatrix} 2/5 & 4/5 & 0 \\ -4/5 & 2/5 & 0 \\ 14/39 & -22/39 & 10/39 \end{bmatrix}$$

By applying $\begin{pmatrix} R_1 \to R_1 + \frac{13}{5} R_3 \\ R_2 \to R_2 - \frac{1}{5} R_3 \end{pmatrix}$, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B^{-1} = \begin{bmatrix} 4/3 & -2/3 & 2/3 \\ -34/39 & 20/39 & -2/39 \\ 14/39 & -22/39 & 10/39 \end{bmatrix}$$

$$\Rightarrow IB^{-1} = B^{-1} = \frac{2}{39} \begin{bmatrix} 26 & -13 & 13 \\ -17 & 10 & -1 \\ 7 & -11 & 5 \end{bmatrix}$$

Sol.12 Given $A = \begin{vmatrix} 2 & 0 & -\alpha \\ 5 & \alpha & 0 \\ 0 & \alpha & 3 \end{vmatrix}$

If A^{-1} exist then $|A| \neq 0$

$\Rightarrow 2(3\alpha - 0) – 0(15 - 0) - \alpha(5\alpha - 0) \neq 0$

$\Rightarrow 6\alpha - 5\alpha^2 \neq 0$

$\Rightarrow \alpha(6 - 5\alpha) \neq 0$

$$\Rightarrow \alpha \neq 0,\ \frac{6}{5}$$

$$\Rightarrow \alpha \in R - \left\{0, \frac{6}{5}\right\}$$

Every square matrix satisfies its charecteristic equation

i.e. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & -\alpha \\ 5 & \alpha-\lambda & 0 \\ 0 & \alpha & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2 - \lambda)\{(\alpha - \lambda)(3 - \lambda)\} - \alpha(5\alpha) = 0$

$\Rightarrow \lambda^3 - (\alpha + 5)\lambda^2 + (5\alpha + 6)\lambda + (5\alpha^2 - 6\alpha) = 0$

$\Rightarrow A^3 - (\alpha + 5)A^2 + (5\alpha + 6)A + (5\alpha^2 - 6\alpha)I = 0$

Multiplying both sides by A^{-1}, we get

$\Rightarrow A^{-1}A^3 - (\alpha+5)A^{-1}A^2 + (5\alpha+6)\ A^{-1}A + (5\alpha^2 - 6\alpha)A^{-1}I$

$= A^{-1}\ 0$

$\Rightarrow A^2 - (\alpha + 5)\ A + (5\alpha + 6)I + (5\alpha^2 - 6\alpha)A^{-1} = 0$

$\Rightarrow A^{-1}\ (\alpha - 5\alpha^2) = A^2 - (\alpha + 5)A + (5\alpha + 6)I$

$$\Rightarrow A^{-1} = \frac{1}{(6\alpha - 5\alpha^2)}\ [A^2 - (\alpha + 5)A + (5\alpha + 6)I]$$

when $\alpha = 1$

$\Rightarrow A^{-1} = A^2 - 6A + 11I$

Sol.13 Given system of equations $3x + 2y + z = 41$

$2x + y + 2z = 29$

$x + y + z = 22$

$$\text{Here } D = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 3(1-2)–2(2-2) + 1(2-1) = -2$$

$$D_1 = \begin{vmatrix} 41 & 2 & 1 \\ 29 & 1 & 2 \\ 22 & 1 & 1 \end{vmatrix} = 41(1-2)–2(29-44) + 1(29-22) = –4$$

$$D_2 = \begin{vmatrix} 41 & 2 & 1 \\ 29 & 1 & 2 \\ 22 & 1 & 1 \end{vmatrix} = 3(29-44) - 41(2-2) + 1(44-29) = -30$$

$$D_3 = \begin{vmatrix} 3 & 2 & 41 \\ 2 & 1 & 29 \\ 1 & 1 & 22 \end{vmatrix} = 3(22-29) - 2(44-29) + 41(2-1) = -10$$

By cramer's rule : $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$

$z = 2,\quad y = 15,\quad z = 5$

Sol.14 **(i)** Given system of equations $2x - y + 3z = 8$

$-x + 2y + z = 4$

$3x + y - 4z = 0$

$$\text{Here } D = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2(–8–1)+1(4–3)+3(–1–6) = –38$$

$$D_1 = \begin{vmatrix} 8 & -1 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -4 \end{vmatrix} = 8(–8–1)+1(–16–0)+3(4–0) = –76$$

$$D_2=\begin{vmatrix}8 & -1 & 3\\ 4 & 2 & 1\\ 0 & 1 & -4\end{vmatrix}=2(-16-0)-8(4-3)+3(0-12)=-76$$

$$D_3=\begin{vmatrix}2 & -1 & 8\\ -1 & 2 & 4\\ 3 & 1 & 0\end{vmatrix}=2(0-4)+1(0-12)+8(-1-6)=-76$$

By cramer's rule : $x=\frac{D_1}{D}$, $y=\frac{D_2}{D}$, $z=\frac{D_3}{D}$

$x = 2$, $y = 2$, $z = 2$

(ii) Given system of equations $x + y + z = 8$

$2x + 5y + 7z = 52$

$2x + y - z = 0$

$$\text{Here } D=\begin{vmatrix}1 & 1 & 1\\ 2 & 5 & 7\\ 2 & 1 & -1\end{vmatrix}=1(-5-7)-1(-2-14)+1(2-10)=-4$$

$$D_1=\begin{vmatrix}8 & 1 & 1\\ 52 & 5 & 7\\ 0 & 1 & -1\end{vmatrix}=8(-5-7)-1(-52-0)+1(52-0)=8$$

$$D_2=\begin{vmatrix}1 & 8 & 1\\ 2 & 52 & 7\\ 2 & 0 & -1\end{vmatrix}=1(-52-0)-8(-2-14)+1(0-104)=-28$$

$$D_3=\begin{vmatrix}1 & 1 & 8\\ 2 & 5 & 52\\ 2 & 1 & 0\end{vmatrix}=1(0-52)-1(0-104)+8(2-10)=-12$$

By cramer's rule : $x=\frac{D_1}{D}$, $y=\frac{D_2}{D}$, $z=\frac{D_3}{D}$

$x = -2$, $y = 7$, $z = 3$

Sol.15 $A=\begin{bmatrix}3 & -2 & 3\\ 2 & 1 & -1\\ 4 & -3 & 2\end{bmatrix}$

$|A| = 3(2-3) + 2(4+4) + 3(-6-4) = (-17)$

$$\text{adj } A=\begin{bmatrix}-1 & -8 & -10\\ -5 & -6 & 1\\ -1 & 9 & 7\end{bmatrix}^T=\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$$

$$A^{-1}=\frac{1}{(-17)}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$$

Given system of equations,

$3x + 3z = 8 + 2y \Rightarrow 3x - 2y + 3z = 8$

$2x + y = 1 + z \quad \Rightarrow 2x + y - z = 1$

$4x + 2z = 4 + 3y \Rightarrow 4x - 3y + 2z = 4$

$$\therefore \begin{bmatrix}3 & -2 & 3\\ 2 & 1 & -1\\ 4 & -3 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}8\\ 1\\ 4\end{bmatrix} \Rightarrow AX = B$$

$\Rightarrow X = A^{-1}B$

$$\Rightarrow X=\frac{1}{(-17)}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}$$

$$\Rightarrow X=\frac{1}{(-17)}\begin{bmatrix}-17\\ -34\\ -51\end{bmatrix} \Rightarrow x = 1;\ y = 2;\ z = 3$$

Sol.16 Given $A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$

Now $A^T=\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$

$$A^{-1}=\frac{\text{adj }A}{|A|}=\frac{1}{1+\tan^2 x}\begin{bmatrix}1 & -\tan x\\ +\tan x & 1\end{bmatrix}$$

$$\text{then } A^T A^{-1}=\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}\frac{1}{1+\tan^2 x}\begin{bmatrix}1 & -\tan x\\ +\tan x & 1\end{bmatrix}$$

$$=\frac{1}{1+\tan^2 x}\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}\begin{bmatrix}1 & -\tan x\\ +\tan x & 1\end{bmatrix}$$

$$=\frac{1}{1+\tan^2 x}\begin{bmatrix}1-\tan^2 x & -2\tan x\\ 2\tan x & 1-\tan^2 x\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1-\tan^2 x}{1+\tan^2 x} & \frac{-2\tan x}{1+\tan^2 x}\\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x}\end{bmatrix}=\begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}$$

Sol.17 Given $A=\begin{bmatrix}0 & 2y & z\\ x & y & -z\\ x & -y & z\end{bmatrix}$ obeys the law $A^T A = I$

Now $A^T A = I$

$$\Rightarrow \begin{bmatrix}0 & x & x\\ 2y & y & -y\\ z & -z & z\end{bmatrix}\begin{bmatrix}0 & 2y & z\\ x & y & -z\\ x & -y & z\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2x^2 & 0 & 0\\ 0 & 6y^2 & 0\\ 0 & 0 & 3z^2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\therefore \quad 2x^2 = 1,\ 6y^2 = 1,\ 3z^2 = 1$

$\therefore \quad x = \pm \frac{1}{\sqrt{2}},\ y = \pm \frac{1}{\sqrt{6}},\ z = \pm \frac{1}{\sqrt{3}}$

Sol.18 Given : $A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$\therefore \quad |A| = -1(-3-1) - 2(2+1) + 5(2-3) = (-7)$

$\&\ \text{adj } A = \begin{bmatrix} -4 & -3 & -1 \\ 3 & 4 & -1 \\ 17 & 11 & -1 \end{bmatrix}^T = \begin{bmatrix} -4 & 3 & 17 \\ -3 & 4 & 11 \\ -1 & -1 & -1 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{(-7)} \begin{bmatrix} -4 & 3 & 17 \\ -3 & 4 & 11 \\ -1 & -1 & -1 \end{bmatrix}$

Given system of equations.

$\begin{bmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ -3 \end{bmatrix}$

$\Rightarrow \quad AX = B$

$\Rightarrow \quad X = A^{-1}B$

$\Rightarrow \quad X = \frac{1}{(-7)} \begin{bmatrix} -4 & 3 & 17 \\ -3 & 4 & 11 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 15 \\ -3 \end{bmatrix}$

$\Rightarrow \quad X = \frac{1}{(-7)} \begin{bmatrix} -14 \\ 21 \\ -14 \end{bmatrix} \Rightarrow x = 2;\ y = (-3);\ z = 2$

Sol.19 Given : $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$

$\therefore \quad |A| = 1(1+6) + 2(2-0) + 0(-4-0)$

$\Rightarrow |A| = 11$

$\text{adj } A = \begin{bmatrix} 7 & -2 & -4 \\ 2 & 1 & 2 \\ -6 & -3 & 5 \end{bmatrix}^T = \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{11} \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$

Given System of equations,

$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

$\Rightarrow AX = B$

$\Rightarrow X = A^{-1}B = \frac{1}{11} \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

$\Rightarrow X = \frac{1}{11} \begin{bmatrix} 44 \\ -33 \\ 11 \end{bmatrix} \Rightarrow x = 4;\ y = (-3);\ z = 1$

Sol.20 Given : $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix};\ B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

Also, $(A + B)^2 = A^2 + B^2$

$\Rightarrow \quad A^2 + B^2 + AB + BA = A^2 + B^2$

$\Rightarrow AB = -BA$

$\Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = -\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} = -\begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$

On comparing corresponding elements

$a - b = -a - 2;$ $\quad\quad 2 = a + 1;$

$2a - b = -b + 2;$ $\quad\quad 3 = b - 1;$

$\Rightarrow a = 1 \quad \& \quad b = 4$

Sol.21 For infinite solutions,

$D = D_1 = D_2 = D_3 = 0$

Now, $D = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{vmatrix}$

$\Rightarrow D = 5(260 - 4) - 3(30 - 14) + 7(6 - 182) = 0$

$\&\ D_1 = \begin{vmatrix} 4 & 3 & 7 \\ 9 & 26 & 2 \\ 5 & 2 & 10 \end{vmatrix}$

$\Rightarrow D_1 = 4(260 - 4) - 3(90 - 10) + 7(18 - 130) = 0$

$\&\ D_2 = \begin{vmatrix} 5 & 4 & 7 \\ 3 & 9 & 2 \\ 7 & 5 & 10 \end{vmatrix}$

$\Rightarrow D_2 = 5(90 - 10) - 4(30 - 14) + 7(15 - 63) = 0$

$\&\ D_3 = \begin{vmatrix} 5 & 3 & 4 \\ 3 & 26 & 9 \\ 7 & 2 & 5 \end{vmatrix}$

$\Rightarrow D_3 = 5(130 - 18) - 3(15 - 63) + 4(6 - 182) = 0$

Sol.22 Given : $\Delta = \begin{vmatrix} \sin\theta & 1 & 0 \\ 1 & \cos\phi & -\cos\theta \\ \sin\phi & 0 & 1 \end{vmatrix}$

$\Rightarrow \Delta = \sin\theta(\cos\phi - 0) - 1(1 + \sin\phi\cos\theta) + 0$
$\Rightarrow \Delta = \sin\theta\cos\phi - \sin\phi\cos\theta - 1$
$\Rightarrow \Delta = \sin(\theta - \phi) - 1$
$\therefore \Delta_{max} = 1 - 1 = 0$
& $\Delta_{min} = -1 - 1 = (-2)$

Sol.23 Given : $\begin{vmatrix} e^x & \sin x \\ \cos x & \ell n(1+x) \end{vmatrix} = A + Bx + Cx^2 + ...$

$\Rightarrow e^x.\ \ell n(1 + x) - \cos x.\sin x = A + Bx + Cx^2 + ...$

$\Rightarrow (1 + x + ...)\left(x - \frac{x^2}{2} + ...\right) - \frac{1}{2}\sin 2x = A + Bx + Cx^2 + ...$

$\Rightarrow (1 + x + ...)\left(x - \frac{x^2}{2} + ...\right) - \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + ...\right)$

$= A + Bx + Cx^2 + ...$
On comparing constant term, A = 0
On comparing coefficient of x, B = 1 – 1 = 0

Sol.24 Given : $\begin{vmatrix} b^2 + c^2 & ab & ac \\ ab & c^2 + a^2 & bc \\ ca & cb & a^2 + b^2 \end{vmatrix}$

$\Rightarrow \frac{1}{abc}\begin{vmatrix} ab^2 + ac^2 & ab^2 & ac^2 \\ a^2b & bc^2 + a^2b & bc^2 \\ ca^2 & cb^2 & a^2c + b^2c \end{vmatrix}$

Applying $C_1 \Rightarrow C_1 - C_2 - C_3$

$\Rightarrow \frac{1}{abc}\begin{vmatrix} 0 & ab^2 & ac^2 \\ -2bc^2 & bc^2 + a^2b & bc^2 \\ -2b^2c & cb^2 & a^2c + b^2c \end{vmatrix}$

$\Rightarrow \frac{abc}{abc}\begin{vmatrix} 0 & b^2 & c^2 \\ -2c^2 & c^2 + a^2 & c^2 \\ -2b^2 & b^2 & a^2 + b^2 \end{vmatrix}$

$\Rightarrow -b^2(-2c^2a^2 - 2b^2c^2 + 2b^2c^2)$
$\quad + c^2(-2c^2b^2 + 2b^2c^2 + 2b^2a^2)$
$\Rightarrow 2a^2b^2c^2 + 2a^2b^2c^2 = 4a^2b^2c^2$

Sol.25 L.H.S. = $\begin{vmatrix} a_1\ell_1 + b_1m_1 & a_1\ell_2 + b_1m_2 & a_1\ell_3 + b_1m_3 \\ a_2\ell_1 + b_2m_1 & a_2\ell_2 + b_2m_2 & a_2\ell_3 + b_2m_3 \\ a_3\ell_1 + b_3m_3 & a_3\ell_2 + b_3m_2 & a_3\ell_3 + b_3m_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix} \times \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ 0 & 0 & 0 \end{vmatrix}$

where $k_1, k_2, k_3 \in R$

$= \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix} \times 0 = 0 = $ R.H.S.

Sol.26 Given equations are $x + y + z = 6$
$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$

Here $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 1(2\lambda - 6) - 1(\lambda - 3) + 1(2 - 2) = \lambda - 3$

$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix} = 2\lambda + \mu - 16$

$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = 2\lambda - \mu + 4$

$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = \mu - 10$

By cramer's rule : $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$

(a) A unique solution : $D \neq 0$ i.e. $\lambda \neq 3$

(b) Infinite number of solutions :
$D = D_1 = D_2 = D_3 = 0$
i.e. $\lambda = 3$, $\mu = 10$

(c) No solution : D = 0 and at least one D_1, D_2, D_3 is not zero
i.e. $\lambda = 3$, $\mu \neq 10$